COMPATIBLE MAPPINGS USING IMPLICIT RELATIONS IN MENGER SPACES

Balbir Singh
Department of Mathematics
B.M.Institute of Engineering and Technology,
Sonipat, Haryana, India.
balbir.vashist007@gmail.com

Abstract. In this paper, we introduce the notions of compatible mappings of type (R), type (K) and type (E) in Menger spaces and prove some common fixed point theorems for these mappings. In fact, we call these maps as variants of compatible mappings.

Mathematics Subject Classification: 47H10, 54H25.

Key Words: Menger space, Compatible mappings, Compatible mappings of type (R), type (K), type (E).

1. Introduction

The notion of probabilistic metric space as a generalization of metric space was introduced by Menger [6]. In Menger theory, the notion of probabilistic metric space corresponds to situations when we do not know exactly the distance between two points, but we know probabilities of possible values of this distance. In this note he explained how to replace the numerical distance between two points \( p \) and \( q \) by a function \( \mathcal{F}(p, q, t) \) whose value \( \mathcal{F}(p, q, t) \) at the real number \( t \) is interpreted as the probability that the distance between \( p \) and \( q \) is less than \( t \). In fact the study of such spaces received an impetus with the pioneering work of Schweizer and Sklar [10]. The theory of probabilistic metric space is of paramount importance in Probabilistic Functional Analysis especially due to its extensive applications in random differential as well as random integral equations.

Now, we give preliminaries and basic definitions in Menger space which are useful in this paper.

Definition 1.1[10] A mapping \( \mathcal{F} : \mathbb{R}^+ \to \mathbb{R}^+ \) is called distribution function if it is non decreasing and left continuous with \( \inf \{ \mathcal{F}(t) : t \in \mathbb{R}^+ \} = 0 \) and \( \sup \{ \mathcal{F}(t) : t \in \mathbb{R}^+ \} = 1 \). We will denote the set of all distribution functions by \( \mathcal{L} \).

Let \( \mathcal{L} \) be the set of all distribution functions whereas \( \mathcal{H} \) be the set of specific distribution function (Also known as Heaviside function) defined by

\[
\mathcal{H}(t) = \begin{cases} 
0, & \text{if } t \leq 0 \\
1, & \text{if } t > 0.
\end{cases}
\]

Definition 1.2[6] A probabilistic metric space is a pair \( (\mathbb{K}, \mathcal{F}) \), where \( \mathbb{K} \) is a nonempty set and \( \mathcal{F} : \mathbb{K} \times \mathbb{K} \to \mathcal{L} \) is a mapping satisfying the following:

For all \( p, q, r \in \mathbb{K} \) and \( t, s \geq 0 \),

\[
\begin{align*}
(p_1) \mathcal{F}(p, q, t) &= 1 \text{ if and only if } p = q; \\
(p_2) \mathcal{F}(p, q, 0) &= 0; \\
(p_3) \mathcal{F}(p, q, t) &= \mathcal{F}(q, p, t); \\
(p_4) \mathcal{F}(p, q, t) &= 1 \text{ and } \mathcal{F}(q, r, s) = 1, \text{ then } \mathcal{F}(p, r, t + s) = 1.
\end{align*}
\]

Every metric space \( (\mathbb{K}, d) \) can always be realized as a Probabilistiv metric space by \( \mathcal{F}(p, q, t) = \mathcal{H}(t - d(p, q)) \), for all \( p, q \in \mathbb{K} \), where \( \mathcal{H} \) be the set of specific distribution function defined in the definition 1.1[10].

Probabilistic metric space offers a wider framework than that of the metric space and cover even wider statistical situations.

Definition 1.3[10] A mapping \( \Delta : [0,1] \times [0,1] \to [0,1] \) is called a \( t \)-norm if for all \( a, b, c \in [0,1] \),

\[
\begin{align*}
(1) \Delta(a, 1) &= a, \Delta(0,0) = 0; \\
(2) \Delta(a, b) &= \Delta(b, a); \\
(3) \Delta(c, d) &\geq \Delta(a, b) \text{ for } c \geq a, d \geq b; \\
(4) \Delta(\Delta(a, b), c) &= \Delta(a, \Delta(b, c)).
\end{align*}
\]

Example 1.4 The following are the four basic \( t \)-norms:

\[
\begin{align*}
(i) \text{ The minimum } t\text{-norm: } \Delta_m(a, b) &= \min\{a, b\}. \\
(ii) \text{ The product } t\text{-norm: } \Delta_p(a, b) &= ab. \\
(iii) \text{ The Lukasiewicz } t\text{-norm: } \Delta_l(a, b) &= \max\{a + b - 1, 0\}. \\
(iv) \text{ The weakest } t\text{-norm, the drastic product: }
\end{align*}
\]
\[ \Delta_D(a,b) = \begin{cases} \min\{a,b\} & \text{if } \max\{a,b\} = 1, \\ 0, & \text{otherwise}. \end{cases} \]

We have the following ordering in the above stated norms:

\[ \Delta_D < \Delta_L < \Delta_P < \Delta_M. \]

**Definition 1.5** [6] A Menger space is a triplet \((\mathbb{K}, \mathcal{F}, \Delta)\), where \((\mathbb{K}, \mathcal{F})\) is a probabilistic metric space and \(\Delta\) is a \(\ell\)-norm with the following condition:

For all \(p, q, r \in \mathbb{K}\) and \(\ell, s \geq 0\),

\[ (p, s) \mathcal{F}(p, r, \ell + s) \geq \Delta(\mathcal{F}(p, q, \ell), \mathcal{F}(q, r, s)). \]

**Example 1.6** Let \(\mathbb{K} = [0, 1]\), \(\Delta(a, b) = \min\{a, b\}\), for all \(a, b \in [0, 1]\) and

\[ \mathcal{F}(p, q) = \{\mathcal{H}(\ell), \text{ if } p \neq q; \text{ where } \mathcal{H}(\ell) = \begin{cases} 0, & \text{if } \ell \leq 0 \\ 1, & \text{if } \ell > 0. \end{cases} \]

Then \((\mathbb{K}, \mathcal{F}, \Delta)\) is a Menger space.

**Definition 1.7** A sequence \(\{p_n\}\) in Menger space \((\mathbb{K}, \mathcal{F}, \Delta)\) is said to be:

(i) Convergent at a point \(p \in \mathbb{K}\) if for every \(\varepsilon > 0\) and \(\lambda > 0\), there exists a positive integer \(N_{\varepsilon, \lambda}\) such that \(\mathcal{F}(p_n, p, \varepsilon) > 1 - \lambda\) for all \(n \geq N_{\varepsilon, \lambda}\).

(ii) Cauchy sequence in \(\mathbb{K}\) if for every \(\varepsilon > 0\) and \(\lambda > 0\), there exists a positive integer \(N_{\varepsilon, \lambda}\) such that \(\mathcal{F}(p_n, p_m, \varepsilon) > 1 - \lambda\) for all \(n, m \geq N_{\varepsilon, \lambda}\).

(iii) Complete if every Cauchy sequence in \(\mathbb{K}\) is convergent in \(\mathbb{K}\).


**Definition 1.8**[4] Two self-mapping \(f_1\) and \(g_1\) of a Menger space \((\mathbb{K}, \mathcal{F}, \Delta)\) are said to be weakly commuting if \(\mathcal{F}(f_1g_1p, q, f_1p, \ell) \geq \mathcal{F}(f_1p, q, p, \ell)\) for each \(p \in \mathbb{K}\) and each \(\ell > 0\).


In 1991, Mishra [7] introduced the notion of compatible mappings in the setting of probabilistic metric space.

**Definition 1.9**[7] Let \((\mathbb{K}, \mathcal{F}, \Delta)\) be a Menger space such that the \(\ell\)-norm \(\Delta\) is continuous and \(f_1, g_1\) be mappings from \(\mathbb{K}\) into itself. Then \(f_1\) and \(g_1\) are said to be compatible if \(\lim_{n \to \infty} \mathcal{F}(f_1g_1p_n, g_1f_1p_n, \ell) = 1\), whenever \(\{p_n\}\) is a sequence in \(\mathbb{K}\) such that \(\lim_{n \to \infty} f_1p_n = \lim_{n \to \infty} g_1p_n = u_1\), for some \(u_1 \in \mathbb{K}\).

**Definition 1.10** Two self-mappings \(f_1\) and \(g_1\) on Menger space \((\mathbb{K}, \mathcal{F}, \Delta)\) are said to be non-compatible if either

\[ \lim_{n \to \infty} \mathcal{F}(f_1g_1p_n, g_1f_1p_n, \ell) \neq 1, \]

whenever \(\{p_n\}\) is a sequence in \(\mathbb{K}\) such that \(\lim_{n \to \infty} f_1p_n = \lim_{n \to \infty} g_1p_n = u_1\), for some \(u_1 \in \mathbb{K}\).

Further, Singh and Jain [12] proved some fixed point theorems for weakly compatible maps in the setting of Menger space.

**Definition 1.11[12]** Two maps \(f_1\) and \(g_1\) are said to be weakly compatible if they commute at their coincidence points.

In 1999, Pant [8] introduced a new continuity condition in Menger space, known as reciprocal continuity as follows:

**Definition 1.12**[8] Let \(f_1\) and \(g_1\) be self-mapping of a Menger space \((\mathbb{K}, \mathcal{F}, \Delta)\). Then \(f_1\) and \(g_1\) are said to be reciprocally continuous if \(\lim_{n \to \infty} f_1g_1p_n = f_1r, \lim_{n \to \infty} g_1f_1p_n = g_1r\), whenever \(\{p_n\}\) is a sequence in \(\mathbb{K}\) such that \(\lim_{n \to \infty} f_1p_n = \lim_{n \to \infty} g_1p_n = r\) for some \(r \in \mathbb{K}\).

**Remark 1.13**[8] If \(f_1\) and \(g_1\) are both continuous, then they are obviously reciprocally continuous, but the converse is not true. Moreover, common fixed point theorems for compatible pair of self mappings satisfying contractive conditions, continuity of one of the mappings implies their reciprocal continuity, but not conversely.

In 2004, Rohan et al. [9] introduced the concept of compatible mappings of type (R) in a metric space as follows:

**Definition 1.14**[9] Let \(f_1\) and \(g_1\) be mappings from metric space \((\mathbb{K}, d)\) into itself. Then \(f_1\) and \(g_1\) are said to be compatible of type (R) if
\[
\lim_{n \to \infty} d(f_1 g_1 p_n, g_1 f_1 p_n) = 0 \quad \text{and} \quad \lim_{n \to \infty} d(f_1 f_1 p_n, g_1 g_1 p_n) = 0,
\]
whenever \( \{p_n\} \) is a sequence in \( \mathbb{K} \) such that \( \lim_{n \to \infty} f_1 p_n = \lim_{n \to \infty} g_1 p_n = u_1 \), for some \( u_1 \) in \( \mathbb{K} \).

In 2007, Singh and Singh et al. [13] introduced the concept of compatible mappings of type (E) in a metric space as follows:

**Definition 1.15**[13] Two self mappings \( f_1 \) and \( g_1 \) of a metric space \( (\mathbb{K}, d) \) are said to be compatible of type (E) if \( \lim_{n \to \infty} f_1 f_1 p_n = \lim_{n \to \infty} g_1 g_1 p_n = g_1 u_1 \) and \( \lim_{n \to \infty} g_1 g_1 p_n = \lim_{n \to \infty} g_1 f_1 p_n = f_1 u_1 \), whenever \( \{p_n\} \) is a sequence in \( \mathbb{K} \) such that \( \lim_{n \to \infty} f_1 p_n = u_1 \) for some \( t \) in \( \mathbb{K} \).

In 2014, Jha et al. [1] introduced the concept of compatible mappings of type (K) in a metric space as follows:

**Definition 1.16**[1] Let \( f_1 \) and \( g_1 \) be mappings from metric space \( (\mathbb{K}, d) \) into itself. Then \( f_1 \) and \( g_1 \) are said to be compatible of type (K) if

\[
\lim_{n \to \infty} d(f_1 f_1 p_n, g_1 u_1) = 0 \quad \text{and} \quad \lim_{n \to \infty} d(g_1 g_1 p_n, f_1 u_1) = 0,
\]

whenever \( \{p_n\} \) is a sequence in \( \mathbb{K} \) such that \( \lim_{n \to \infty} f_1 p_n = \lim_{n \to \infty} g_1 p_n = u_1 \), for some \( u_1 \) in \( \mathbb{K} \).

2. **Properties of variants of compatible mappings.**

Now we present the notions of variants of compatible mappings in the context of a Menger space.

**Definition 2.1** Let \( S \) and \( T \) are two self-mapping on Menger space \( (\mathbb{K}, \mathcal{F}, \Delta) \). Then \( S \) and \( T \) are said to be:

1. **Compatible of type (R)** if \( \lim_{n \to \infty} F(ST x_n, TS x_n, t_1) = 1 \), and

\[
\lim_{n \to \infty} F(SS x_n, TT x_n, t_1) = 1,
\]

whenever a sequence \( \{x_n\} \) in \( \mathbb{K} \) satisfying \( \lim_{n \to \infty} S x_n = \lim_{n \to \infty} T x_n = u_1 \), where \( u_1 \in \mathbb{K}, \forall \ t_1 > 0 \).

2. **Compatible of type (K)** if \( \lim_{n \to \infty} F(SS x_n, SU_1, t_1) = 1 \) and

\[
\lim_{n \to \infty} F(TT x_n, SU_1, t_1) = 1,
\]

whenever a sequence \( \{x_n\} \) in \( \mathbb{K} \) satisfying \( \lim_{n \to \infty} S x_n = \lim_{n \to \infty} T x_n = u_1 \), where \( u_1 \in \mathbb{K} \).

3. **Compatible of type (E)** if \( \lim_{n \to \infty} S S x_n = \lim_{n \to \infty} S T x_n = \lim_{n \to \infty} T U_1 \) and

\[
\lim_{n \to \infty} F(ST x_n, SU_1, t_1) = \lim_{n \to \infty} F(TS x_n, SU_1, t_1) = 1.
\]

Therefore, \( \lim_{n \to \infty} S T x_n = U_1 \).

**Proposition 2.1** Let \( S \) and \( T \) are two compatible mappings of type (R) self maps of a Menger space \( (\mathbb{K}, \mathcal{F}, \Delta) \). If \( u_1 = T u_1 \), for some \( u_1 \) in \( \mathbb{K} \), then \( SU_1 = TS U_1 = TT U_1 = TS U_1 \).

**Proof** Suppose that \( \{x_n\} \) is a sequence in \( \mathbb{K} \) defined by \( x_n = u_1, n = 1, 2, ... \) for some \( u_1 \) in \( \mathbb{K} \) and \( S u_1 = T u_1 \). Then we have \( S x_n, T x_n \rightarrow S u_1 \) as \( n \rightarrow \infty \). Since \( S \) and \( T \) are compatible of type (R), we have

\[
F(ST U_1, TS U_1, t_1) = \lim_{n \to \infty} F(ST x_n, TS x_n, t_1) = 1.
\]

Hence we have \( ST u_1 = SS U_1 \). Therefore, since \( S U_1 = T U_1 \), we have \( ST U_1 = SS U_1 = TT U_1 = TS U_1 \).

**Proposition 2.2** Let \( S \) and \( T \) are two compatible mappings of type (R) self maps of Menger space \( (\mathbb{K}, \mathcal{F}, \Delta) \). Consider that \( \lim_{n \to \infty} S x_n = \lim_{n \to \infty} T x_n = u_1 \), where \( u_1 \) in \( \mathbb{K} \). Then

(a) \( \lim_{n \to \infty} TS x_n = S u_1 \) if \( S \) is continuous at \( u_1 \).

(b) \( \lim_{n \to \infty} ST x_n = T u_1 \) if \( T \) is continuous at \( u_1 \).

(c) \( ST u_1 = TS U_1 \) and \( S u_1 = T u_1 \) if \( S \) and \( T \) are continuous at \( u_1 \).

**Proof** (a) Suppose that \( S \) is continuous at \( u_1 \). Since \( \lim_{n \to \infty} S x_n = \lim_{n \to \infty} T x_n = u_1 \) where \( u_1 \) in \( \mathbb{K} \), we have \( SS x_n, ST x_n \rightarrow S u_1 \) as \( n \rightarrow \infty \). Then by given condition

\[
\lim_{n \to \infty} F(TS x_n, SU_1, t_1) = \lim_{n \to \infty} F(TS x_n, ST x_n, t_1) = 1.
\]

Therefore, \( \lim_{n \to \infty} TS x_n = S u_1 \).

(b) Suppose that \( T \) is continuous at \( u_1 \). Since \( \lim_{n \to \infty} S x_n = \lim_{n \to \infty} T x_n = u_1 \), where \( u_1 \) in \( \mathbb{K} \), we have \( TS x_n, TT x_n \rightarrow T u_1 \) as \( n \rightarrow \infty \). Then by given condition

\[
\lim_{n \to \infty} F(ST x_n, SU_1, t_1) = \lim_{n \to \infty} F(ST x_n, TS x_n, t_1) = 1.
\]

Therefore, \( \lim_{n \to \infty} ST x_n = T u_1 \).
3. MAIN RESULTS

Recently, Kang et al. [10] proved the some common fixed point theorem in a complete multiplicative metric space. Now we prove the same in Menger space using implicit relations.

Theorem 3.1 Let \( f, g, f_1 \) and \( g_1 \) be mappings of a complete Menger space \((\mathbb{K}, \mathcal{F}, \Delta)\) into itself satisfying the following conditions:

\[
\begin{align*}
(3.1) & \quad g_1(\mathbb{K}) \subset f(\mathbb{K}), \quad f_1(\mathbb{K}) \subset g(\mathbb{K}); \\
(3.2) & \quad \psi \left( \mathcal{F}(f, p, q, k), \mathcal{F}(f, p, q, t), \mathcal{F}(f, p, q, t), \mathcal{F}(f, p, q, (1 + k)t) \right) \geq 1 \\
& \quad \text{for all } p, q \in \mathbb{K}, \text{ where } k \in (0, 1), \psi \in \mathbb{K}, t > 0.
\end{align*}
\]

(3.3) one of the mappings \( f, g, f_1 \) and \( g_1 \) is continuous.

Assume that the pairs \( f, f_1 \) and \( g, g_1 \) are compatible of type (R). Then \( f, g, f_1 \) and \( g_1 \) have a unique common fixed point in \( \mathbb{K} \).

Proof Since \( f_1(\mathbb{K}) \subset g(\mathbb{K}) \). Now consider a point \( p_0 \in \mathbb{K} \), there exists a \( p_1 \in \mathbb{K} \) such that \( f_1 p_0 = g p_1 = q_0 \) for this point \( p_1 \) there exists a \( p_2 \in \mathbb{K} \) such that \( g_1 p_1 = f p_2 = q_1 \). Continuing in this way, we can define a sequence \( \{q_n\} \) in \( \mathbb{K} \) such that

\[
q_{2n} = f q_{2n-1} = g p_{2n+1}; \quad q_{2n+1} = g_1 p_{2n+1} = f p_{2n+2}.
\]

Now we prove that \( \{q_n\} \) is Cauchy sequence in \( \mathbb{K} \).

Putting \( p = p_{2n} \), \( q = p_{2n+1} \) in inequality (3.2), we have

\[
\psi \left( \mathcal{F}(f, p_{2n}, q, k), \mathcal{F}(f, p_{2n}, q, t), \mathcal{F}(f, p_{2n}, f p_{2n+1}), \mathcal{F}(f, p_{2n+1}, f p_{2n+2}), \mathcal{F}(f, p_{2n+2}, f p_{2n+3}), \mathcal{F}(f, p_{2n+3}, (1 + k)t) \right) \geq 1
\]

Since the function \( \psi \) is non-increasing in the 6th coordinate variable. Using properties of implicit relations \( \mathcal{F} \), we get

\[
\mathcal{F}(q_{2n}, q_{2n+1}, k) \geq \mathcal{F}(q_{2n-1}, q_{2n+1}, k)
\]

Again putting \( p = p_{2n+1} \), \( q = p_{2n+2} \) in inequality (3.2), we have

\[
\psi \left( \mathcal{F}(f, p_{2n+1}, g q_{2n+2}, k), \mathcal{F}(f, p_{2n+1}, g q_{2n+2}, t), \mathcal{F}(f, p_{2n+2}, g q_{2n+2}, t), \mathcal{F}(f, p_{2n+2}, g q_{2n+2}, (1 + k)t) \right) \geq 1
\]
We claim that $\psi$ is non-increasing in the $6^\text{th}$ coordinate variable. Using properties of implicit relations, we get
\[
\mathcal{F}(q_{2n+1}, q_{2n+2}, k t) \geq \mathcal{F}(q_{2n}, q_{2n+1}, t)
\]
Thus for all $n \in \mathbb{N}$ and $t > 0$, we have
\[
\mathcal{F}(q_{n}, q_{n+1}, k t) \geq 
\]
Therefore, by Lemma 2.1, $\{q_n\}$ is a Cauchy sequence in $\mathbb{K}$ and hence it converges to some point $r \in K$. Consequently, the subsequence $\{f_k p_{2n}\}$, $\{g_k p_{2n}\}$, $\{f_k p_{2n+1}\}$ and $\{f_k p_{2n+2}\}$ of $\{q_n\}$ also converges to $r$.

Now, suppose that $f$ is continuous. Since $f$ and $f_1$ are compatible of type (R), by Proposition 2.1, $f f p_{2n}$ and $f_1 f p_{2n}$ converges to $fr$ as $n \to \infty$.

We claim that $r = fr$.

Putting $p = f p_{2n}$ and $q = p_{2n+1}$, in inequality (3.2) we have
\[
1 \leq \psi \left( \mathcal{F}(fr, r, kt), \mathcal{F}(fr, r, t), \mathcal{F}(fr, f r, t), \mathcal{F}(fr, f r, t), \mathcal{F}(fr, fr, t), \mathcal{F}(fr, fr, t) \right)
\]
Using properties of implicit relations, we get
\[
\mathcal{F}(fr, r, kt) \geq \mathcal{F}(fr, r, t).
\]
By Lemma 2.2, we $fr = r$.

Next we claim that $r = f_1 r$.

Putting $p = r$ and $q = p_{2n+1}$, in inequality (3.2) we have
\[
1 \leq \psi \left( \mathcal{F}(f_1 r, r, p_{2n+1}, k t), \mathcal{F}(f_1 r, r, p_{2n+1}, t), \mathcal{F}(f_1 r, f_1 r, t), \mathcal{F}(fr, fr, t), \mathcal{F}(fr, fr, t) \right) \geq 1
\]
Letting $n \to \infty$ we have
\[
\psi \left( \mathcal{F}(f_1 r, r, k t), \mathcal{F}(r, r, t), \mathcal{F}(r, f_1 r, t), \mathcal{F}(fr, fr, t), \mathcal{F}(fr, fr, t) \right) \geq 1
\]
Using properties of implicit relations, we get
\[
\mathcal{F}(f_1 r, r, kt) \geq \mathcal{F}(fr, r, t).
\]
By Lemma 2.2, we $f_1 r = r$.

Since $f_1(\mathbb{K}) \subset g(\mathbb{K})$ and hence exists a point $u_1 \in \mathbb{K}$ such that $r = f_1 r = g u_1$.

We claim that $r = g u_1$.

Putting $p = r$ and $q = u_1$, in inequality (3.2) we have
\[
1 \leq \psi \left( \mathcal{F}(f_1 r, g_1 u_1, k t), \mathcal{F}(fr, g u_1, t), \mathcal{F}(fr, fr, t), \mathcal{F}(fr, fr, t) \right)
\]
Using properties of implicit relations, we get
\[
\mathcal{F}(fr, r, kt) \geq \mathcal{F}(fr, r, t).
\]
By Lemma 2.2, we $fr = r$. Since $f_1(\mathbb{K}) \subset g(\mathbb{K})$ and hence exists a point $v_1 \in \mathbb{K}$ such that $r = f_1 r = g v_1$.

We claim that $r = g v_1$. 

Vol. 2, Issue 1, June 2018
Putting $p = f_1 p_{2n}$ and $q = v_1$ in inequality (3.2), we have
\[
1 \leq \psi \left( \mathcal{F}(f_1 p_{2n} g_1 v_1, t), \mathcal{F}(f_1 p_{2n} g_1 v_1, t), \mathcal{F}(f_1 p_{2n} f_1 p_{2n}, t) \right)
\]
Letting $n \to \infty$ we have
\[
1 \leq \psi \left( \mathcal{F}(r, g_1 u_1, t), \mathcal{F}(r, r, t), \mathcal{F}(r, r, t), \mathcal{F}(r, r, t) \right)
\]
Using properties of implicit relations $\mathcal{I}$, we get
\[
\mathcal{F}(r, r, t) \geq \mathcal{F}(r, r, t)
\]
By Lemma 2.2, we get $r = g_1 u_1$. Since $g$ and $g_1$ are compatible of type (R) and $g v_1 = g_1 v_1 = r$, by Proposition 2.1, $g g_1 v_1 = g_1 g v_1$ and hence $gr = g g_1 v_1 = g_1 g v_1 = g_1 r$.

We claim that $r = g_1 r$.

Putting $p = p_{2n}$ and $q = r$ in inequality (3.2) we have
\[
1 \leq \psi \left( \mathcal{F}(f_1 p_{2n} g_1 r, t), \mathcal{F}(f_1 p_{2n} g_1 r, t), \mathcal{F}(f_1 p_{2n} f_1 p_{2n}, t) \right)
\]
Letting $n \to \infty$ we have
\[
1 \leq \psi \left( \mathcal{F}(r, g_1 r, t), \mathcal{F}(r, g_1 r, t), \mathcal{F}(r, g_1 r, t) \right)
\]
Using properties of implicit relations $\mathcal{I}$, we get
\[
\mathcal{F}(r, g_1 r, t) \geq \mathcal{F}(r, g_1 r, t)
\]
By Lemma 2.2, we get $g_1 r = r$.

Since $f$ and $f_1$ are compatible of type (R) and $f_1 w = f w = r$, by Proposition 2.1, $f f_1 w = f_1 f w$ and hence $fr = f f_1 w = f_1 f w = f_1 r$.

Hence $r = gr = g_1 r = fr = f_1 r$. Therefore, $r$ is a common fixed point of $f, f_1, g$ and $g_1$.

Similarly, we can complete the proof when $g_1$ is continuous.

Next we prove the following theorem for compatible mappings of type (K) and type (E).

**Theorem 3.2** Let $f, g, f_1$ and $g_1$ be mappings of a complete Menger space $(\mathbb{K}, \mathcal{F}, \Delta)$ into itself satisfying the following conditions (3.1), (3.2). Suppose that the pairs $f, f_1$ and $g, g_1$ are reciprocally continuous.

Assume that the pairs $f, f_1$ and $g, g_1$ are compatible of type (K). Then $f, f_1, g$ and $g_1$ have a unique common fixed point in $\mathbb{K}$.

**Proof** Now from the proof of Theorem 3.1 we can easily prove that $\{q_n\}$ is Cauchy sequence in $\mathbb{K}$ and hence it converges to some point $r \in \mathbb{K}$. Consequently, the subsequences $\{f_1 p_{2n}\}$, $\{g p_{2n+1}\}$ and $\{f p_{2n}\}$ of $\{q_n\}$ also converges to $r$.

Since the pairs $f, f_1$ and $g, g_1$ are compatible of type (K), we have $f f_1 p_{2n} \to f_1 r, f f_1 p_{2n} \to f r$ and $g p_{2n} \to g_1 r, g_1 g p_{2n+1} \to g r$ as $n \to \infty$.

We claim that $gr = fr$.

Putting $p = f_1 p_{2n}$ and $q = g_1 p_{2n+1}$ in inequality (3.2) we have
\[
1 \leq \psi \left( \mathcal{F}(f_1 p_{2n} g_1 p_{2n+1}, t), \mathcal{F}(f_1 p_{2n} g_1 p_{2n+1}, t), \mathcal{F}(f_1 p_{2n} f_1 p_{2n}, t) \right)
\]
Letting $n \to \infty$ and using reciprocity of continuity of the pairs $f, f_1$ and $g, g_1$ we have
\[
1 \leq \psi \left( \mathcal{F}(fr, gr, kt), \mathcal{F}(fr, fr, t), \mathcal{F}(fr, fr, t), \mathcal{F}(fr, fr, (1+k)t) \right)
\]
Using properties of implicit relations $\mathcal{I}$, we get
\[
\mathcal{F}(fr, gr, kt) \geq \mathcal{F}(fr, gr, kt)
\]
By Lemma 2.2, we get $fr = gr$.

Next we claim that $gr = f_1 r$.

Putting $p = r$ and $q = g_1 p_{2n+1}$ in inequality (3.2) we have
\[
1 \leq \psi \left( \mathcal{F}(fr, g_1 p_{2n+1}, t), \mathcal{F}(fr, f_1 p_{2n+1}, t), \mathcal{F}(fr, fr, t), \mathcal{F}(fr, fr, t) \right)
\]
Using properties of implicit relations $\mathcal{I}$, we get
\[
\mathcal{F}(fr, g_1 p_{2n+1}, t) \geq \mathcal{F}(fr, fr, t)
\]
By Lemma 2.2, we get $fr = g_1 r$.

Similarly, we can complete the proof when $g_1$ is continuous.
Letting \( n \to \infty \) and using reciprocal continuity of the pairs \( f, f_1 \) and \( g, g_1 \) we have

\[
1 \leq \psi \left( F(f_1 r, g r, k t), F( r, g r, t), F(g r, k r, t), F( f_1 r, g r, t), F(g r, k r, (1 + k) t) \right)
\]

Using properties of implicit relations \( \mathcal{I} \), we get

\[
F(f_1 r, g r, t) \geq F(r, g r, k t).
\]

By Lemma 2.2, we get \( f_1 r = g r \).

We claim that \( f_1 r = g_1 r \).

Putting \( p = r \) and \( q = r \) in inequality (3.2) we have

\[
1 \leq \psi \left( F(r, g r, k t), F( f_1 r, g r, r, t), F(f_1 r, g r, t), F(r, g r, (1 + k) t) \right)
\]

Using properties of implicit relations \( \mathcal{I} \), we get

\[
F(f_1 r, g r, t) \geq F(r, g r, k t).
\]

By Lemma 2.2, we get \( f_1 r = g_1 r \).

Putting \( p = p_{2n} \) and \( q = r \) in inequality (3.2) we have

\[
1 \leq \psi \left( F(f_1 r, g_1 r, k t), F(p_{2n}, g r, t), F(p_{2n}, f_1 r, t), F(p_{2n}, g r, (1 + k) t) \right)
\]

Letting \( n \to \infty \), we have

\[
1 \leq \psi \left( F(r, g r, k t), F( r, g r, r, t), F(r, g r, t), F(r, g r, (1 + k) t) \right)
\]

Using properties of implicit relations \( \mathcal{I} \), we get

\[
F(r, g r, t) \geq F(r, g r, k t).
\]

By Lemma 2.2, we get \( r = g_1 r \).

Hence \( r = g r = g_1 r = f_1 r \). Therefore, \( r \) is a common fixed point of \( f, f_1, g \) and \( g_1 \).

**Theorem 3.3** Let \( f, f_1 \) and \( g_1 \) be mappings of a complete Menger space \( (\mathbb{K}, F, \Delta) \) into itself satisfying the following conditions (3.1), (3.2). Suppose that one of \( f \) and \( f_1 \) is continuous and one of \( g \) and \( g_1 \) is continuous.

Assume that the pairs \( f, f_1 \) and \( g, g_1 \) are compatible of type (E). Then \( f, g, f_1 \) and \( g_1 \) have a unique common fixed point in \( \mathbb{K} \).

**Proof** Now from the proof of Theorem 3.1 we can easily prove that \( \{q_n\} \) is Cauchy sequence in \( \mathbb{K} \) and hence it converges to some point \( r \in \mathbb{K} \). Consequently, the subsequence \( \{f_{1p_{2n}}\}, \{g_{p_{2n+1}}\}, \{g_{p_{2n+1}}\} \) and \( \{f_{p_{2n}}\} \) of \( \{q_n\} \) also converges to \( r \).

Now, suppose that one of the mappings \( f \) and \( f_1 \) is continuous. Since \( f \) and \( f_1 \) are compatible of type (E), by Proposition 2.3, \( f_1 r = f_1 r \). Since \( f_1(K) \subset g(K) \) and hence exists a point \( w \in K \) such that \( f_1 r = g w \).

We claim that \( f_1 r = g_1 w \).

Putting \( p = r \) and \( q = w \) in inequality (3.2) we have

\[
1 \leq \psi \left( F(f_1 r, g_1 w, k t), F(r, g w, t), F(r, f_1 r, t), F(g w, g w, t) \right).
\]

Using properties of implicit relations \( \mathcal{I} \), we get

\[
F(f_1 r, g_1 w, t) \geq F(r, g_1 w, (1 + k) t).
\]

By Lemma 2.2, we get \( f_1 r = g_1 w \). Thus we have \( f_1 r = g_1 w = g w \).

We claim that \( f_1 r = r \).

Putting \( p = r \) and \( q = p_{2n} \) in inequality (3.2) we have

\[
1 \leq \psi \left( F(f_1 r, g_1 p_{2n+1}, k t), F(r, g p_{2n+1}, r, t), F(r, f_1 r, t), F(r, f_1 r, t), F(r, g p_{2n+1}, t) \right)
\]

Using properties of implicit relations \( \mathcal{I} \), we get

\[
F(f_1 r, g_1 p_{2n+1}, k t) \geq F(r, g p_{2n+1}, (1 + k) t).
\]

By Lemma 2.2, we get \( f_1 r = g_1 p_{2n+1} \). Therefore, \( r \) is a common fixed point of \( f, f_1, g \) and \( g_1 \).

Again, suppose \( g \) and \( g_1 \) are compatible of type (E) and one of the mappings \( g \) and \( g_1 \) is continuous. Then we get \( g w = g_1 w = r \). By Proposition 2.3, we have \( g w = g_1 w = g_1 g w = g_1 g_1 w \). Hence \( g w = g_1 r \).

We claim that \( r = g_1 r \).

37
Putting $p = p_{2n}$ and $q = r$ in inequality (3.2) we have
\[ 1 \leq \psi \left( \frac{\mathcal{F}(f_{1}p_{2n}, g_{1}v, k_{t})}{\mathcal{F}(f_{1}p_{2n}, g_{1}r, k_{t})}, \frac{\mathcal{F}(f_{1}p_{2n}, g_{1}v, k_{t})}{\mathcal{F}(f_{1}p_{2n}, g_{1}r, k_{t})} \right) \]

Letting $n \to \infty$ we have
\[ 1 \leq \psi \left( \frac{\mathcal{F}(f_{1}r, r, k_{t})}{\mathcal{F}(f_{1}r, r, k_{t})} \right) \]

Using properties of implicit relations $\mathcal{I}$, we get
\[ \mathcal{F}(f_{1}r, r, k_{t}) \geq \mathcal{F}(f_{1}r, r, k_{t}) \]

By Lemma 2.2, we get $r = g_{1}v$. Since $g$ and $g_{1}$ are compatible of type (R) and $g_{1}v = g_{1}r = r$, by Proposition 2.2. $g_{1}v = g_{1}v$ and hence $gr = g_{1}v = g_{1}v = g_{1}r$. We claim that $r = g_{1}r$.

Putting $p = p_{2n}$ and $q = r$, in inequality (3.2) we have
\[ 1 \leq \psi \left( \frac{\mathcal{F}(f_{1}p_{2n}, g_{1}r, k_{t})}{\mathcal{F}(f_{1}p_{2n}, g_{1}r, k_{t})}, \frac{\mathcal{F}(f_{1}p_{2n}, g_{1}r, k_{t})}{\mathcal{F}(f_{1}p_{2n}, g_{1}r, k_{t})} \right) \]

Letting $n \to \infty$ we have
\[ 1 \leq \psi \left( \frac{\mathcal{F}(f_{1}r, g_{1}r, k_{t})}{\mathcal{F}(f_{1}r, g_{1}r, k_{t})}, \frac{\mathcal{F}(f_{1}r, g_{1}r, k_{t})}{\mathcal{F}(f_{1}r, g_{1}r, k_{t})} \right) \]

Using properties of implicit relations $\mathcal{I}$, we get
\[ \mathcal{F}(f_{1}r, g_{1}r, k_{t}) \geq \mathcal{F}(f_{1}r, g_{1}r, k_{t}) \]

REFERENCES


(2) G. Jungck, Commuting mappings and fixed points, Amer. Math. Mon., 83(1976), 261-263.


(11) S. Sessa, On a weak commutativity condition of mappings in fixed point considerations, Publications De’ Institute Mathematic, Nouvelle Serie Tome, 32(1982), 149-153.
